Asymptotic behavior of the condition number of two-level Toeplitz matrix sequences

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Abstract

In this note we give condition number estimates for two-level Toeplitz matrices generated by bivariate 2π-periodic weakly sectorial symbols (the largest class of symbols for which we can guarantee the invertibility of all the corresponding Toeplitz matrices). By using and extending tools previously developed in the one-level context, we show that the asymptotical ill-conditioning is essentially related to the order of zeros of the symbol.

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1 Introduction

The aim of this paper is the analysis of the Euclidean conditioning $\kappa(\cdot)$ (i.e. with respect to the spectral norm = the largest singular value) of two-level Toeplitz matrices of the form $T_{nm}(f)$ where n and m are large and where the symbol f is assumed to be L^∞ over $I^2 = (-\pi, \pi]^2$ and weakly sectorial. We recall that a function is weakly sectorial if and only if there exists a straight line s passing from the complex zero such that its essential range is all contained in one of the two closed half planes having s as frontier and it is not all contained in s (see [1]): following a simple reasoning (a complex rotation of the essential range), it is evident that we can reduce the above reasoning to the case where the real part of f is nonnegative and not identically zero and with generic imaginary part.

In a more explicit language the entries of $T_{nm}(f)$ can be described as follows: $(T_{nm}(f))_{(j,k)(p,q)} = t_{k-j,q-p}$ with $t_{r,s}$ being the Fourier coefficients of f, i.e.,

$$t_{r,s} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-\mathrm{i}(rx+sy)} dx dy, \quad \mathrm{i}^2 = -1.$$

Here the 2-index notation $(T_{nm}(f))_{(j,k)(p,q)}$ indicates that we are selecting the block (j,k) of size m with $j,k \in \{1,\ldots,n\}$ and, in that block, we are selecting the entry (p,q), $p,q \in \{1,\ldots,m\}$.

Such a kind of matrices (often also called block Toeplitz with Toeplitz blocks) arise in several applications (see e.g. [2]) such as Markov chains, integral equations, in the solution of certain partial differential equations (PDEs), image restoration etc and for these applications the solution of large linear systems is often required so that the study of the conditioning of the related problem is a key point.

Indeed, for estimating the inherent error in the solution of a corresponding linear system and in understanding the convergence speed of iterative methods, it is of crucial interest to evaluate precisely the asymptotic behavior of the condition numbers of $T_{nm}(f)$: this is in fact the topic of the present paper.

More precisely we will show that the upper bound of the condition number will depend on the maximal order of the zeros of $\mathrm{Re} f$: the presence of a large $\mathrm{Im}\, f$ will decrease the condition number. Tight lower estimates can be found when f is real valued i.e. the imaginary part is identically zero and more generally in the case where the zeros of the imaginary part include those of the real part with equal or higher orders. Our analysis includes the simple case of a finite number of isolated zeros and the more involved situation of a finite number of smooth curves of zeros as well.

The paper is organized in two more sections: in Section 2 we report all the theoretical results and we give some practical examples; Section 3 contains a discussion on the results and some conclusive remarks.

2 Condition number estimation for two-level Toeplitz matrices

Let $f \in L^\infty$ be a 2-variate, 2π-periodic function and suppose $u(x,y) \geq 0$ where $u(x,y) = \mathrm{Re} f(x,y)$ and $v(x,y) = \mathrm{Im}\ f(x,y)$. Assume that u has a finite number of distinct zeros $(x_j, y_j) \in I^2 = (-\pi,\pi]^2$, $j = 1,\ldots,k$, has infinitely many zeros which form a finite set of disjoint smooth curves $\mathcal{C}_j := \{(x,y) :\ C_j(x,y) = 0\}$, $j = 1,\ldots,q$, with $C_j(x,y)$ regular enough and 2π-periodic, and suppose that u is positive elsewhere. For $Y \subset I^2$, we define by $S(Y,\delta) = \cup_{(\bar{x},\bar{y})\in Y}\{(x,y) : \|(x,y) - (\bar{x},\bar{y})\|_\infty < \delta\}$. A point $(\bar{x},\bar{y}) \in I^2$ is said to be a zero of $u \geq 0$ if for $Y = \{(\bar{x},\bar{y})\}$ we have

$$\operatorname{ess\,inf}\{u(x,y) : (x,y) \in S(Y,\delta)\} = 0 \quad \forall\, \delta > 0;$$

moreover $(\bar{x},\bar{y}) \in I^2$ is a distinct zero if it is a zero and there exists a value $\delta_1 > 0$ such that

$$\operatorname{ess\,inf}\{u(x,y) : (x,y) \in S(Y,\delta_1)\backslash S(Y,\delta)\} > 0 \quad \forall \delta :\ 0 < \delta < \delta_1.$$

In the same lines $\overline{\mathcal{C}}$ is said to be a curve of zeros of u if every point of $\overline{\mathcal{C}}$ is a zero and $\overline{\mathcal{C}}$ is a curve; moreover $\overline{\mathcal{C}}$ is a isolated curve of zeros if, setting $Y = \overline{\mathcal{C}}$, there exists a value $\delta_1 > 0$ such that

$$\operatorname{ess\,inf}\{u(x,y) : (x,y) \in S(Y,\delta_1)\backslash S(Y,\delta)\} > 0 \quad \forall \delta :\ 0 < \delta < \delta_1.$$

Then

$$\operatorname{ess\,inf} u > 0 \text{ on } I^2 \setminus S(Y,\delta)$$

for all $\delta > 0$ and where $Y = \left\{\cup_{j=1}^{k}\{(x_j,y_j)\}\right\} \cup \left\{\cup_{j=1}^{q}\mathcal{C}_j\}\right\}$ is the set of all the essential zeros of u. We fix now $\delta > 0$ so that the domains $S(Y,\delta)$ are pairwise disjoint so that Y is either a isolated zero $\{(x_j,y_j)\}$, $j \in \{1,\ldots,k\}$ or a isolated curve of zeros $\mathcal{C}_j$, $j \in \{1,\ldots,q\}$. Then we define the functions

$\omega_{(\bar{x},\bar{y})}$ and $\omega_{\overline{C}}$ for $(\bar{x},\bar{y})$ being an essential zero of u and $\overline{C}$ being an isolated curve of zeros of u:

$$\frac{1}{\omega_{(\bar{x},\bar{y})}(\nu)} = \inf_{\substack{\{R_k\}_{k\in K},\, \#K<\infty,\ R_k \text{ circular sector} \\ \text{with positive angle independent of } \delta \\ \cup_k R_k = S(\{(\bar{x},\bar{y})\},\delta)}} \min_k \sup_{\substack{c(t)\subset R_k \\ t\in[0,1]}} \left\{ \operatorname{ess\,inf}\left\{ u(x,y) : \frac{1}{\nu} < \|(x,y)-(\bar{x},\bar{y})\|_\infty < \delta, (x,y)\in c(t) \right\}\right\}, \tag{1}$$

where $\delta > 0$, $(\bar{x},\bar{y})$ is the unique zero of u in $S(\{(\bar{x},\bar{y})\},\delta)$, $t \in [0,1]$, and $c(t)$ ranges among all possible curves in $I^2 \cap R_k$ intersecting the point $(\bar{x},\bar{y})$.

$$\omega_{\overline{C}}(\nu) = \sup_{(\bar{x},\bar{y})\in\overline{C}} \omega_{(\bar{x},\bar{y})}(\nu). \tag{2}$$

These functions characterize the order of an isolated zero and of an isolated curve of zeros respectively.

Definition 2.1 Let $(\bar{x},\bar{y})$ be a isolated zero of u; its order is defined as

$$\lim_{\nu\to\infty} \frac{\log(\omega_{(\bar{x},\bar{y})}(\nu))}{\log(\nu)}.$$

According to the same lines we define the order of the zeros forming a isolated curve $\overline{C}$ of zeros.

For instance for

$$f(x,y) = x^2 + (x+y^2)^4$$

we have $\omega_{(0,0)}(\nu) \sim \nu^2$ and for

$$f(x,y) = |1-x^2-y^2| + (x^2+|y|)^4\, e^{-\frac{1}{\sqrt{|1-x^2-y^2|}}}$$

we have $\omega_C(\nu) \sim \nu$ with C being the circle defined by $x^2+y^2=1$.

In an analogous way we can define zeros or curves of zeros of logarithmic orders and the ones of exponential orders. The first case happens when the quantity in Definition 2.1 is equal to zero: in this situation we should talk of zero order which includes the case of a zero of logarithmic type as

$$f(x,y) = \frac{1}{|\log(10^{-1}(|x|+|y|))|}$$

with $\omega_{(0,0)}(\nu) \sim \log(\nu)$. The second situation occurs when the quantity considered in Definition 2.1 is equal to $+\infty$: in this case we should talk of zero of ∞ order (or of super polynomial type) which includes the case of a zero of exponential type as

$$f(x,y) = e^{-\frac{1}{x^2+|y|^3}}$$

with $\omega_{(0,0)}(\nu) \sim e^{\nu^2}$ and as

$$f(x,y) = \begin{cases} e^{-\frac{1}{\sqrt{|x-1|+|y-1|}}} & x > 1, y > 1 \\ (x-1)^2 + (y-x)^4 & \text{elsewhere} \end{cases}$$

with $\omega_{(1,1)}(\nu) \sim e^{\sqrt{\nu}}$.

The given notion of order of zero stated in Definition 2.1 is very important for studying the asymptotical ill-conditioning of matrices $T_{nm}(f)$ with f weakly sectorial and $n, m \to \infty$.

As an example we consider the real valued symbol $x^2 + y^2$. It is simple to show that $4 - 2\cos(x) - 2\cos(y) \le x^2 + y^2 \le \pi^2/2[4 - 2\cos(x) - 2\cos(y)]$ and therefore by the linear positivity of the Toeplitz operators (see [3]) we have

$$\lambda_j(\Delta) \le \lambda_j(T_{nm}(f)) \le \pi^2 \lambda_j(\Delta)/2$$

where $\lambda_j(X)$ indicates the j-th eigenvalue (in nondecreasing order) of the Hermitian matrix X and $\Delta = T_{nm}(4 - 2\cos(x) - 2\cos(y))$ is the discretized Laplacian with zero boundary conditions on a 2 dimensional rectangle. The eigenvalues of Δ are explicitly known and in particular we have

$$\lambda_{\min}(\Delta) \sim \pi^2(n^{-2} + m^{-2})/4,$$

$$\lim_{n,m\to\infty} \lambda_{\max}(\Delta) = 8.$$

Therefore the Euclidean condition number of $T_{nm}(f)$ is asymptotic to $(n^{-2} + m^{-2})^{-1}$ which is bounded by $\omega(n+m) \sim (n+m)^2$ and is asymptotic to it if $n \sim m$. The latter fact is very general and holds for the larger class of weakly sectorial symbols. Indeed the subsequent upper estimate result for the condition number is true.

2.1 An estimate from above

Theorem 2.1 Let $f \in L^\infty$ be a 2-variate, 2π-periodic function. Suppose $\mathrm{Re}\ f \ge 0$, and assume $u = \mathrm{Re}\ f$ has a finite number of distinct zeros

$(x_j, y_j) \in I^2 = (-\pi, \pi]^2$, $j = 1, \ldots, k$, has infinitely many zeros which form a finite set of disjoint curves $\mathcal{C}_j := \{(x, y) : C_j(x, y) = 0\}$, $j = 1, \ldots, q$, with $C_j(x, y)$ regular enough and 2π-periodic, and suppose that u is positive elsewhere. Define $\omega_j(n) = \omega_{(x_i, y_i)}(n)$, $i = 1, \ldots, k$, and $\omega_{C_j}(n)$, $j = 1, \ldots, q$, by Eqs. (1) and (2) respectively, put

$$\omega(n) := \max_{\substack{1 \le i \le k \\ 1 \le j \le q}} \{\omega_i(n), \omega_{C_j}(n)\},$$

and let $v = \mathrm{Im} f$. If $\forall j = 1, \ldots, q$ we have

$$\max \left\{ \sup_{r \in \mathcal{H}} \#(\mathcal{C}_j \cap r), \sup_{r \in \mathcal{V}} \#(\mathcal{C}_j \cap r) \right\} < \infty, \tag{3}$$

with $\mathcal{H}$ denoting the set of all the horizontal lines and with $\mathcal{V}$ denoting the set of all the vertical lines, then

$$\kappa(T_{nm}(f)) \le 12\|f\|_\infty(\|v\|_\infty + 1)\omega(c(n + m)), \tag{4}$$

for all sufficiently large n, m, where $T_{nm}(f)$ is the $(nm \times nm)$ two-level block Toeplitz matrix generating by the function f and $c > 1$ is an absolute constant.

Proof: We consider the discretization $-\pi = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_N = \pi$ along the x axis and we define the sets $\bar{X}_i = \{(x, y) : x \in (\bar{x}_{i-1}, \bar{x}_i], y \in (-\pi, \pi], i = 1, \cdots, N\}$, which are stripes along the y direction. Suppose that the set $\bar{X}_i$ contains some of the distinct zeros and some disjoint pieces of some of the curves of zeros. We consider now all the distinct zeros and just one point for every piece of the curve, to form a set of k_i points (x_{i_j}, y_{i_j}), $j = 1, \ldots, k_i$. We fix n, m and use the classical result by Dirichlet (Lemma 3.5 of Böttcher and Grudsky [1]), with $\mu = 1/12$, $\beta_j = (nx_{i_j} + my_{i_j})/(2\pi)$, $j = 1, 2, \ldots, k_i$, to get an integer q_i such that

$$1 \le q_i \le 13^{k_i},\ q_i(nx_{i_j} + my_{i_j}) \in 2\pi Z + \left(-\frac{\pi}{6}, \frac{\pi}{6}\right),\ j = 1, 2, \cdots, k_i. \tag{5}$$

We follow the same trigonometric manipulations as in the proof of Theorem 3.4 of Böttcher and Grudsky in [1]:

$$\begin{aligned} \cos(q_i(nx + my)) &= \cos(q_i(nx_{i_j} + my_{i_j}))\cos(q_i(n(x - x_{i_j}) + m(y - y_{i_j}))) \\ &- \sin(q_i(nx_{i_j} + my_{i_j}))\sin(q_i(n(x - x_{i_j}) + m(y - y_{i_j}))). \end{aligned} \tag{6}$$

From (5) we have

$$\cos(q_i(nx_{i_j} + my_{i_j})) > \frac{\sqrt{3}}{2}, \sin(q_i(nx_{i_j} + my_{i_j})) < \frac{1}{2}, j = 1, \dots, k_i.$$

We choose the points $(x, y) \in \bar{X}_i$ close enough to (x_{i_j}, y_{i_j}), such that

$$q_i(n(x - x_{i_j}) + m(y - y_{i_j})) \le q_i(n + m)\|(x, y) - (x_{i_j}, y_{i_j})\|_\infty < \frac{\pi}{6}$$

or equivalently $\|(x, y) - (x_{i_j}, y_{i_j})\|_\infty < \dfrac{\pi}{6q_i(n + m)}$. Then

$$\cos(q_i(n(x - x_{i_j}) + m(y - y_{i_j}))) > \frac{\sqrt{3}}{2}, \sin(q_i(n(x - x_{i_j}) + m(y - y_{i_j}))) < \frac{1}{2}$$

and by using the latter inequalities in (6), we find

$$\cos(q_i(nx + my)) > \frac{1}{2}. \tag{7}$$

¿From the regularity of the curves of zeros and since q_i is integer, if we replace a point (x_{i_j}, y_{i_j}) (from the set of k_i points) with another belonging to the same piece of the same curve, then we get the same integer q_i from the Dirichlet result. Thus we come back to construct the discretization by choosing the successive points $\bar{x}_{i-1}, \bar{x}_i$ to be so close each other such that the integer q_i corresponding to $\bar{X}_i$ will be independent of the choice $(x_{i_j}, y_{i_j}) \in \bar{X}_i$. The distance $\bar{x}_i - \bar{x}_{i-1}$ can be estimated as follows.
If there exists a piece of curve in $\bar{X}_i$ with endpoints $(\bar{x}_{i-1}, \bar{y}_{i-1})$ and $(\bar{x}_i - \bar{y}_i)$ this piece must be transformed, via (5), into $(-\frac{\pi}{6}, \frac{\pi}{6})$. So, $q_i(n(\bar{x}_i - \bar{x}_{i-1}) + m(\bar{y}_i - \bar{y}_{i-1})) < \frac{\pi}{3}$ and by assuming that $\bar{x}_i - \bar{x}_{i-1} \sim \bar{y}_i - \bar{y}_{i-1}$, we obtain

$$\bar{x}_i - \bar{x}_{i-1} = \mathcal{O}\left(\frac{\pi}{3q_i(n + m)}\right). \tag{8}$$

The case where $\bar{x}_i - \bar{x}_{i-1} = o(\bar{y}_i - \bar{y}_{i-1})$, which means that the curve is parallel or tangent to the direction y, is covered as follows: if the curve is parallel by taking the discretization along the axis y, while if it is tangent by taking small enough the distance $\bar{x}_i - \bar{x}_{i-1}$ such that the length of each piece of curve is of order $\mathcal{O}\left(\frac{\pi}{3q_i(n+m)}\right)$. We consider now the integer

$$q = \max_{1 \le i \le N} \{q_i\}$$

and the function

$$g_{nm}(x,y) = \begin{cases} \cos(q_1(nx+my)), & (x,y) \in \bar{X}_1, \\ \cos(q_2(nx+my)), & (x,y) \in \bar{X}_2, \\ \vdots & \\ \cos(q_N(nx+my)), & (x,y) \in \bar{X}_N. \end{cases}$$

We observe that q is bounded by a pure constant independent of n and m since $q_i \leq 13^{k_i}$ and all the k_i's are uniformly bounded from above by an absolute constant not depending on n neither on m thanks to assumption (3).

We put now

$$\frac{1}{\epsilon_j} = 3(\|v\|_\infty + 1)\omega_j \left(\frac{6q(n+m)}{\pi}\right), \quad j = 1, 2, \ldots, k, \tag{9}$$

for each distinct root $(x_j, y_j), j = 1, \ldots, k$,

$$\frac{1}{\epsilon_C} = 3(\|v\|_\infty + 1)\omega_{C_j} \left(\frac{6q(n+m)}{\pi}\right) \quad , \tag{10}$$

for the curve of roots C_j, $j = 1, \ldots, q$, and

$$M = 2(\|v\|_\infty + 1). \tag{11}$$

We consider the function

$$b_{nm}(x,y) = f(x,y) + \mathrm{i}Mg_{nm}(x,y). \tag{12}$$

The Fourier coefficients c_{l_1,l_2} of $\cos(q_{nm}(nx+my))$, $|l_1| \leq n-1$ and $|l_2| \leq m-1$ respectively, are given by

$$\begin{aligned} c_{l_1,l_2} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_{nm}(x,y)e^{-\mathrm{i}(l_1x+l_2y)}dydx \\ &= \sum_{i=1}^{N}\int_{\bar{x}_{i-1}}^{\bar{x}_i}\int_{-\pi}^{\pi}\cos(q_i(nx+my)e^{-\mathrm{i}(l_1x+l_2y)}dydx \\ &= \sum_{i=1}^{N}\int_{\bar{x}_{i-1}}^{\bar{x}_i}\int_{-\pi}^{\pi} e^{\mathrm{i}((q_in-l_1)x+(q_im-l_2)y)} + e^{-\mathrm{i}((l_1+q_in)x+(l_2+q_im)y)}dydx \\ &= \sum_{i=1}^{N}\int_{\bar{x}_{i-1}}^{\bar{x}_i} e^{\mathrm{i}(q_in-l_1)x}dx \cdot \int_{-\pi}^{\pi} e^{\mathrm{i}(q_im-l_2)y}dy \\ &+ \sum_{i=1}^{N}\int_{\bar{x}_{i-1}}^{\bar{x}_i} e^{-\mathrm{i}(l_1+q_in)x}dx \cdot \int_{-\pi}^{\pi} e^{-\mathrm{i}(l_2+q_im)y}dy = 0, \end{aligned}$$

since

$$\int_{-\pi}^{\pi} e^{\mathrm{i}(q_im-l_2)y}dy = 0, \quad \int_{-\pi}^{\pi} e^{-\mathrm{i}(l_2+q_im)y}dy = 0.$$

Therefore $T_{nm}(b_{nm}) = T_{nm}(f)$. We choose n and m so large that $\frac{\pi}{6q(n+m)} < \delta$. We claim now that the essential range of the restriction of b_{nm} to $S_j(\delta) \equiv S(\{(x_j, y_j)\}, \delta)$ lies above the straight line given by Im $z = 1 - \frac{1}{\epsilon_j}$Re z and that the essential range of the restriction of b_{nm} to $S(\mathcal{C}_j, \delta)$, lies above the straight line given by Im $z = 1 - \frac{1}{\epsilon_C}$Re z. Since

$$b_{nm}(x, y) = u(x, y) + \mathrm{i}(v(x, y) + Mg_{nm}(x, y))),$$

we have to prove that

$$v(x, y) + Mg_{nm}(x, y) > 1 - \frac{1}{\epsilon_j} u(x, y)$$

for almost all $(x, y) \in S_j(\delta)$. We prove that actually

$$\frac{1}{\epsilon_j} u(x, y) + Mg_{nm}(x, y) > 1 + \|v\|_\infty, \tag{13}$$

when (x, y) lies in the aforementioned sets. To prove (13) we take the following cases:

i) $\|(x, y) - (x_j, y_j)\|_\infty < \frac{\pi}{6q(n+m)}$. Let that $(x_j, y_j) \in \bar{X}_i$ is a distinct root. We suppose, without loss of generality, that $S_j\left(\frac{\pi}{6q(n+m)}\right) \in \bar{X}_i$, otherwise, since of (8), we redefine the constant q such that each of $S_j\left(\frac{\pi}{6q(n+m)}\right)$, $j = 1, \ldots, k$, should belong in just one of the $\bar{X}_i$'s. Then from relation (9) and the nonnegativity of $u(x, y)$ we get

$$\frac{1}{\epsilon_j} u(x, y) + Mg_{nm}(x, y) \geq M \cos(q_i(nx + my)) > \frac{M}{2} = 1 + \|v\|_\infty.$$

In the case where $(x_j, y_j) \in \mathcal{C}_s$ for some $s = 1, \ldots, q$, since we have chosen q as the largest value of q_i's we have that

$$g_{nm}(x, y) = \cos(q_i(nx + my)) > \frac{1}{2}, (x, y) \in S\left(\mathcal{C}_j, \frac{\pi}{6q(n+m)}\right) \bigcap \bar{X}_i$$

for all $i = 1, \ldots, N$. So

$$\frac{1}{\epsilon_{C_s}} u(x, y) + Mg_{nm}(x, y) > \frac{M}{2} = 1 + \|v\|_\infty.$$

ii) $\frac{\pi}{6q(n+m)} < \|(x,y)-(x_j,y_j)\|_\infty < \delta$. From the definition (1) we obtain

$$\frac{1}{\epsilon_j}u(x,y) + Mg_{nm}(x,y) \geq \frac{1}{\epsilon_j\omega_j(\frac{6q(n+m)}{\pi})} - M = 1 + \|v\|_\infty,$$

for the distinct roots while for the curve of roots C_s, definition (2) gives us

$$\frac{1}{\epsilon_{C_s}}u(x,y) + Mg_{nm}(x,y) \geq \frac{1}{\epsilon_{C_s}\omega_{C_s}(\frac{6q(n+m)}{\pi})} - M = 1 + \|v\|_\infty.$$

At this point, we consider the value

$$\epsilon = \min_{1\leq i\leq k, 1\leq j\leq q}\left\{\epsilon_i, \epsilon_{C_j}\right\}. \tag{14}$$

This corresponds to

$$\omega\left(\frac{6q(n+m)}{\pi}\right) = \max_{1\leq i\leq k, 1\leq j\leq q}\max\{\omega_i, \omega_{C_j}\}\left(\frac{6q(n+m)}{\pi}\right). \tag{15}$$

Thus, the essential range of the restriction of the function b_{nm} to the set $Y = \left\{\cup_{j=1}^k\{(x_j,y_j)\}\right\} \cup \left\{\cup_{j=1}^q C_j\}\right\}$ lies above the line

$$\text{Im } z = 1 - \frac{1}{\epsilon}\text{Re } z. \tag{16}$$

This is true since ϵ is chosen as the smallest value from the ones corresponding to the distinct points as well as along the curve. Since Re $b_{nm} \geq 0$ we introduce the value η given by

$$\eta := \operatorname{ess\,inf}\left\{u(x,y) : (x,y) \in I^2\backslash S(Y,\delta)\right\},$$
$$Y = \left\{\cup_{j=1}^k\{(x_j,y_j)\}\right\} \cup \left\{\cup_{j=1}^q C_j\}\right\}, \quad \delta > 0$$

which is positive.

¿From now on, the prof of Theorem 3.4 by Bötther and Grudsky [1] follows exactly the same with ϵ in the place ϵ_n. Finally we get

$$\begin{aligned}\|T_{nm}^{-1}(f)\| &< \tfrac{4}{\epsilon} = 12(\|v\|_\infty + 1)\omega\left(\tfrac{6q(n+m)}{\pi}\right) \\ &= 12(\|v\|_\infty + 1)\omega(c(n+m))\end{aligned}$$

which completes the proof of the Theorem. •

2.2 Comments on the assumptions of Theorem 2.1

We make two main observations.

- For simplicity we have assumed that the curves of roots are pairwise disjoint. We have to comment here that if there exist curves intersecting each other, then the results of the theorem remain unchanged. The difference in the proof is that we have some sets of intersecting pieces of curves in the band $\bar{X}_i$, instead of some pieces of curves. We consider the same topology by considering small enough stripes such that the corresponding integer q_i will be the same for each choice of points belonging to the set of intersecting pieces.

- It was noted in the proof that if the curve is parallel to the axis y then the discretization is taking along the axis y. This generates now the question: is the considered theorem true in the case where there are at least two curves of roots in which the first is parallel to the axis x and the other is parallel to the axis y? The answer is yes. This case is covered by taking the discretization parallel to an appropriate direction of the form $x+ry$ to which any of the curve of roots is never parallel. Such a direction exists because of the constant number of the curves of roots. Then, by using the 2π-periodic property and some technical integration properties, we obtain $c_{l_1,l_2} = 0$ so that the proof stands in this case too.

In conclusion Theorem 2.1 is true under much more general assumptions concerning the curves of roots.

2.3 An estimate from below

Just for completeness we should recall that Theorem 2.1 has to be combined with the universal bound stated in [4] and proved in [6]. More precisely under the assumption that f is weakly sectorial we know that there exist positive constants C and γ such that $\kappa(T_{nm}(f)) \leq Ce^{\gamma(n+m)}$. Therefore if $e^{\gamma(n+m)} = o(\omega(c(n+m))$ for every $c > 0$ then the estimate in Theorem 2.1 cannot be tight

On the other hand, the upper estimate given in the former Theorem is tight in the sense that there exist functions satisfying the related assumptions for which the estimate is asymptotically sharp, but it is also true that we can construct examples for which the given estimate is not tight at all even if $\omega(c(n+m) = O(e^{\gamma(n+m)})$ for some positive c and γ.

An example of functions belonging to the first class is $f(x,y) = x^2 + y^2$ or $f(x,y) = \mathrm{i}(x^2+y^2)^\alpha + x^2 + y^2$, $\alpha \geq 1$ for which $\kappa(T_{nm}(f)) \sim (n^{-2} + m^{-2})^{-1} \sim \omega(n+m)$: in actuality the estimates are sharp when the zeros of the imaginary part include those of the real part with equal or higher orders. On the other hand if we take the symbol $f(x,y) = \mathrm{i} + x^2 + y^2$ then $\omega(n+m) \sim (n^{-2} + m^{-2})^{-1}$ but $\kappa(T_{nm}(f)) \sim 1$.

The previous examples show that the behavior of the imaginary part plays a role at least in the lower estimate of the condition number as stated and proved in the following Theorem.

Theorem 2.2 Let $\alpha_1, \alpha_2, \beta_1, \beta_2$, be positive numbers, let $(x_0, y_0) \in I^2$, and suppose $f \in L^\infty$ be a 2-variate, 2π-periodic function.

(a) If $f(x,y) = \mathcal{O}(|x-x_0|^{\alpha_1} + |y-y_0|^{\alpha_2})$ as $(x,y) \rightarrow (x_0,y_0)$ then there is a constant $C \in (0,\infty)$ such that

$$\kappa(T_{nm}(f)) \geq C \frac{n^{\alpha_1} m^{\alpha_2}}{n^{\alpha_1} + m^{\alpha_2}} \text{ for all } n \geq 1, m \geq 1.$$

(b) If $f(x,y) = \mathcal{O}(1/|\log|(x-x_0)/\pi||^{\alpha_1} + 1/|\log|(y-y_0)/\pi||^{\alpha_2})$ as $(x,y) \rightarrow (x_0,y_0)$ then there is a constant $C \in (0,\infty)$ such that

$$\kappa(T_{nm}(f)) \geq C \frac{(\log n)^{\alpha_1} (\log m)^{\alpha_2}}{(\log n)^{\alpha_1} + (\log m)^{\alpha_2}} \text{ for all } n \geq 1, m \geq 1.$$

(c) If $f(x,y) = \mathcal{O}(e^{-\beta_1|x-x_0|^{-\alpha_1}} + e^{-\beta_2|y-y_0|^{-\alpha_2}})$ as $(x,y) \rightarrow (x_0,y_0)$ then

$$\lim_{n,m\rightarrow\infty} n^{-k_1} m^{-k_2} \kappa(T_{nm}(f)) = \infty$$

for every $k_1, k_2 > 0$.

Proof: Without loss of generality we suppose that $(x_0,y_0) = (0,0)$. Let us assume that $|a(x,y)| \leq K(|x|^{\alpha_1} + |y|^{\alpha_2})$ for $|x| < \delta$ and $|y| < \delta$ and fix $n, m > 1/\delta$. We consider now the trigonometric polynomials

$$P^{j_1 j_2}_{m_1 m_2}(x,y) = P^{j_1}_{m_1}(x) P^{j_2}_{m_2}(y)$$

where

$$P^j_m(\theta) = (1 + e^{\mathrm{i}\theta} + \cdots + e^{\mathrm{i}m\theta})^j = e^{\mathrm{i}j\theta/2} \left(\frac{\sin\frac{m+1}{2}\theta}{\sin\frac{\theta}{2}} \right)^j$$

as defined by Böttcher and Grudsky in [1]. It is obvious that

$$\begin{aligned}
\|P^{j_1 j_2}_{m_1 m_2}(x,y)\|_2^2 &= \tfrac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|P^{j_1}_{m_1}(x)|^2|P^{j_2}_{m_2}(y)|^2 dxdy \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi}|P^{j_1}_{m_1}(x)|^2 dx \cdot \tfrac{1}{2\pi}\int_{-\pi}^{\pi}|P^{j_2}_{m_2}(y)|^2 dy \\
&= \|P^{j_1}_{m_1}(x)\|_2^2\|P^{j_2}_{m_2}(y)\|_2^2.
\end{aligned}$$

This allows us to find a lower bound of $\|P^{j_1 j_2}_{m_1 m_2}\|_2$ by using Lemma 4.2 of [1] and more specifically we have

$$\|P^{j_1 j_2}_{m_1 m_2}\|_2^2 > \frac{256}{81\pi^2}\frac{1}{\sqrt{j_1 j_2}}(m_1+1)^{2j_1-1}(m_2+1)^{2j_2-1}. \tag{17}$$

We follow now the same technique indicated by Böttcher and Grudsky [1] for the proof of Theorem 4.1. Consider

$$\begin{aligned}
4\pi^2\|fP^{j_1 j_2}_{m_1 m_2}\|_2^2 &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^2\left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1}\left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dxdy \\
&=: \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Phi(x,y)dxdy
\end{aligned}$$

for every m_1, m_2, j_1 and j_2 positive. We fix j_1 and j_2 such that $j_1 > \alpha_1 + 1/2$ and $j_2 > \alpha_2 + 1/2$ and since $\|P^{j_1 j_2}_{m_1 m_2}\|_\infty = (m_1+1)^{j_1}(m_2+1)^{j_2}$ we get

$$\begin{aligned}
\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Phi dxdy &= \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}}\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\Phi dxdy + \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}}\int_{\frac{1}{m_1}<|x|<\delta}\Phi dxdy \\
&+ \int_{\frac{1}{m_2}<|y|<\delta}\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\Phi dxdy + \int_{\frac{1}{m_2}<|y|<\delta}\int_{\frac{1}{m_1}<|x|<\delta}\Phi dxdy \\
&+ \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}}\int_{\delta<|x|<\pi}\Phi dxdy + \int_{\delta<|y|<\pi}\int_{-\frac{1}{m_1}}^{\frac{1}{m_1}}\Phi dxdy \\
&+ \int_{\frac{1}{m_2}<|y|<\delta}\int_{\delta<|x|<\pi}\Phi dxdy + \int_{\delta<|y|<\pi}\int_{\frac{1}{m_2}<|x|<\delta}\Phi dxdy \\
&+ \int_{\delta<|y|<\pi}\int_{\delta<|x|<\pi}\Phi dxdy \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.
\end{aligned}$$

Consequently we have to find upper bounds for the above integrals:

$$\begin{aligned}
I_1 &\leq \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} K^2(|x|^{\alpha_1} + |y|^{\alpha_2})^2 \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dxdy \\
&= K^2 \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} |y|^{2\alpha_2} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dy \cdot \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} dx \\
&+ K^2 \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dy \cdot \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} |x|^{2\alpha_1} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} dx \\
&+ 2K^2 \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} |y|^{\alpha_2} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dy \cdot \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} |x|^{\alpha_1} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} dx \\
&\leq K^2 \frac{1}{m_2^{2\alpha_2}} (m_2+1)^{2j_2} \frac{2}{m_2} \cdot (m_1+1)^{2j_1} \frac{2}{m_1} \\
&+ K^2 (m_2+1)^{2j_2} \frac{2}{m_2} \cdot \frac{1}{m_1^{2\alpha_1}} (m_1+1)^{2j_1} \frac{2}{m_1} \\
&+ 2K^2 \frac{1}{m_2^{\alpha_2}} (m_2+1)^{2j_2} \frac{2}{m_2} \cdot \frac{1}{m_1^{\alpha_1}} (m_1+1)^{2j_1} \frac{2}{m_1} \\
&= K^2 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}}\right)^2 (m_1+1)^{2j_1} (m_2+1)^{2j_2} \frac{4}{m_1 m_2}.
\end{aligned}$$

$$\begin{aligned}
I_2 &\leq \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \int_{\frac{1}{m_1} < |x| < \delta} K^2(|x|^{\alpha_1} + |y|^{\alpha_2})^2 \\
&\quad \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dxdy \\
&= K^2 \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} |y|^{2\alpha_2} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dy \cdot 2\int_{\frac{1}{m_1}}^{\delta} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} dx \\
&+ K^2 \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dy \cdot 2\int_{\frac{1}{m_1}}^{\delta} x^{2\alpha_1} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} dx \\
&+ 2K^2 \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} |y|^{\alpha_2} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dy 2\int_{\frac{1}{m_1}}^{\delta} x^{\alpha_1} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} dx \\
&\leq K^2 \frac{1}{m_2^{2\alpha_2}} (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 2\int_{\frac{1}{m_1}}^{\delta} (\pi/x)^{2j_1} dx \\
&+ K^2 (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 2\int_{\frac{1}{m_1}}^{\delta} x^{2\alpha_1} (\pi/x)^{2j_1} dx \\
&+ 2K^2 \frac{1}{m_2^{\alpha_2}} (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 2\int_{\frac{1}{m_1}}^{\delta} x^{\alpha_1} (\pi/x)^{2j_1} dx \\
&= K^2 \frac{1}{m_2^{2\alpha_2}} (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 2\pi^{2j_1} m_1^{2j_1-1} \\
&+ K^2 (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 22\frac{\pi^{2j_1}}{m_2^{2\alpha_2}} m1^{2j_1-1} \\
&+ 2K^2 \frac{1}{m_2^{\alpha_2}} (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 2\frac{\pi^{2j_1}}{m_1^{\alpha_1}} m1^{2j_1-1} \\
&= 2K^2 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}}\right)^2 m_1^{2j_1-1} (m_2+1)^{2j_2} \frac{2}{m_2} \\
&\leq M_2 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}}\right)^2 m_1^{2j_1-1} m_2^{2j_2-1},
\end{aligned}$$

where $M_2 < \infty$. For the same reason we have

$$I_3 \leq M_3 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}} \right)^2 m_1^{2j_1-1} m_2^{2j_2-1},$$

where $M_3 < \infty$ and

$$\begin{aligned}
I_4 &\leq \int_{\frac{1}{m_2}<|y|<\delta} \int_{\frac{1}{m_1}<|x|<\delta} K^2(|x|^{\alpha_1} + |y|^{\alpha_2})^2 \\
&\quad \left(\frac{\sin \frac{m_1+1}{2}x}{\sin \frac{x}{2}} \right)^{2j_1} \left(\frac{\sin \frac{m_2+1}{2}y}{\sin \frac{y}{2}} \right)^{2j_2} dxdy \\
&= K^2 2 \int_{\frac{1}{m_2}}^{\delta} y^{2\alpha_2} \left(\frac{\sin \frac{m_2+1}{2}y}{\sin \frac{y}{2}} \right)^{2j_2} dy \cdot 2 \int_{\frac{1}{m_1}}^{\delta} \left(\frac{\sin \frac{m_1+1}{2}x}{\sin \frac{x}{2}} \right)^{2j_1} dx \\
&+ K^2 2 \int_{\frac{1}{m_2}}^{\delta} \left(\frac{\sin \frac{m_2+1}{2}y}{\sin \frac{y}{2}} \right)^{2j_2} dy \cdot 2 \int_{\frac{1}{m_1}}^{\delta} x^{2\alpha_1} \left(\frac{\sin \frac{m_1+1}{2}x}{\sin \frac{x}{2}} \right)^{2j_1} dx \\
&+ 2K^2 2 \int_{\frac{1}{m_2}}^{\delta} y^{\alpha_2} \left(\frac{\sin \frac{m_2+1}{2}y}{\sin \frac{y}{2}} \right)^{2j_2} dy \cdot 2 \int_{\frac{1}{m_1}}^{\delta} x^{\alpha_1} \left(\frac{\sin \frac{m_1+1}{2}x}{\sin \frac{x}{2}} \right)^{2j_1} dx \\
&\leq K^2 2 \int_{\frac{1}{m_2}}^{\delta} y^{2\alpha_2} (\pi/y)^{2j_2} dy \cdot 2 \int_{\frac{1}{m_1}}^{\delta} (\pi/x)^{2j_1} dx \\
&+ K^2 2 \int_{\frac{1}{m_2}}^{\delta} (\pi/y)^{2j_2} dy \cdot 2 \int_{\frac{1}{m_1}}^{\delta} x^{2\alpha_1} (\pi/x)^{2j_1} dx \\
&+ 2K^2 2 \int_{\frac{1}{m_2}}^{\delta} y^{\alpha_2} (\pi/y)^{2j_2} dy \cdot 2 \int_{\frac{1}{m_1}}^{\delta} x^{\alpha_1} (\pi/x)^{2j_1} dx \\
&= K^2 \frac{1}{m_2^{2\alpha_2}} 2\pi^{2j_2} m_2^{2j_2-1} \cdot 2\pi^{2j_1} m_1^{2j_1-1} \\
&+ K^2 2\pi^{2j_2} m_2^{2j_2-1} \cdot 2 \frac{\pi^{2j_1}}{m_1^{2\alpha_1}} m1^{2j_1-1} \\
&+ 2K^2 \frac{1}{m_2^{\alpha_2}} 2\pi^{2j_2} m_2^{2j_2-1} \cdot 2 \frac{\pi^{2j_1}}{m_1^{\alpha_1}} m1^{2j_1-1} \\
&= 4K^2 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}} \right)^2 \pi^{2(j_1+j_2)} m_1^{2j_1-1} m_2^{2j_2-1} \\
&\leq M_4 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}} \right)^2 m_1^{2j_1-1} m_2^{2j_2-1},
\end{aligned}$$

where $M_4 < \infty$. Moreover

$$\begin{aligned}
I_5 &\leq \|f\|_\infty \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \int_{\delta<|x|<\pi} \left(\frac{\sin \frac{m_1+1}{2}x}{\sin \frac{x}{2}} \right)^{2j_1} \left(\frac{\sin \frac{m_2+1}{2}y}{\sin \frac{y}{2}} \right)^{2j_2} dxdy \\
&= \|f\|_\infty \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \left(\frac{\sin \frac{m_2+1}{2}y}{\sin \frac{y}{2}} \right)^{2j_2} dy \cdot 2 \int_{\delta}^{\pi} \left(\frac{\sin \frac{m_1+1}{2}x}{\sin \frac{x}{2}} \right)^{2j_1} dx \\
&\leq \|f\|_\infty (m_2+1)^{2j_2} \frac{2}{m_2} \cdot 2 \int_{\delta}^{\pi} (\pi/x)^{2j_1} dx \leq M_5 m_2^{2j_2-1},
\end{aligned}$$

with $M_5 < \infty$. By using the same arguments, we deduce that

$$I_6 \leq M_6 m_1^{2j_1-1},$$

where $M_6 < \infty$ and

$$\begin{aligned} I_7 &\leq \|f\|_\infty \int_{\frac{1}{m_2}<|x|<\delta} \int_{\delta<|x|<\pi} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dxdy \\ &\leq \|f\|_\infty 2\int_{\frac{1}{m_2}}^{\delta} (\pi/y)^{2j_2} dy \cdot 2\int_\delta^\pi (\pi/x)^{2j_1} dx \\ &\leq \|f\|_\infty 2\pi^{2j_2} m_2^{2j_2-1} \cdot 2\int_\delta^\pi (\pi/x)^{2j_1} dx \leq M_7 m_2^{2j_2-1}, \end{aligned}$$

where $M_7 < \infty$. Finally by following an analogous reasoning, we find

$$I_8 \leq M_8 m_1^{2j_1-1},$$

where $M_8 < \infty$ and

$$\begin{aligned} I_9 &\leq \|f\|_\infty \int_{\delta<|y|<\pi} \int_{\delta<|x|<\pi} \left(\frac{\sin\frac{m_1+1}{2}x}{\sin\frac{x}{2}}\right)^{2j_1} \left(\frac{\sin\frac{m_2+1}{2}y}{\sin\frac{y}{2}}\right)^{2j_2} dxdy \\ &\leq \|f\|_\infty 2\int_\delta^\pi (\pi/y)^{2j_2} dy \cdot 2\int_\delta^\pi (\pi/x)^{2j_1} dx = M_9, \end{aligned}$$

where $M_9 < \infty$. Since $j_1 > \alpha_1 + 1/2$ and $j_2 > \alpha_2 + 1/2$, the above bounds of I_5, I_6, I_7 and I_8 are less than of the ones of I_1, I_2, I_3 and I_4, and in addition the same is true for M_9 since it is an absolute constant. Thus by adding all the above inequalities we get

$$4\pi^2 \|fP_{m_1m_2}^{j_1j_2}\|_2^2 \leq M\left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}}\right)^2 m_1^{2j_1-1} m_2^{2j_2-1}, \tag{18}$$

where $M < \infty$. By using the inequality (17) we obtain

$$\|fP_{m_1m_2}^{j_1j_2}\|_2^2 \leq C_1\left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}}\right)^2 \|P_{m_1m_2}^{j_1j_2}\|_2^2, \tag{19}$$

for some constant C_1. Given n and m we write $n = m_1j_1 + k_1$ and $m = m_2j_2 + k_2$, $k_1 \in \{1, \ldots, j_1\}, k_2 \in \{1, \ldots, j_2\}$. From inequality (19) we infer

$$\begin{aligned} \|T_{nm}(f)P_{m_1m_2}^{j_1j_2}\|_2^2 &\leq \|fP_{m_1m_2}^{j_1j_2}\|_2^2 \leq C_1\left(\frac{(2j_1)^{\alpha_1}}{(2j_1m_1)^{\alpha_1}} + \frac{(2j_2)^{\alpha_2}}{(2j_2m_2)^{\alpha_2}}\right)^2 \|P_{m_1m_2}^{j_1j_2}\|_2^2 \\ &\leq C_2\left(\frac{1}{n^{\alpha_1}} + \frac{1}{m^{\alpha_2}}\right)^2 \|P_{m_1m_2}^{j_1j_2}\|_2^2 \\ &\leq C_2\left(\frac{1}{n^{\alpha_1}} + \frac{1}{m^{\alpha_2}}\right)^2 \|T_{nm}^{-1}(f)\|_2^2 \|T_{nm}(f)P_{m_1m_2}^{j_1j_2}\|_2^2, \end{aligned}$$

which implies that

$$\|T_{nm}^{-1}(f)\|_2 \geq C\frac{n^{\alpha_1}m^{\alpha_2}}{n^{\alpha_1} + m^{\alpha_2}},$$

and the part (a) of the theorem is proven.

b) For the proof of the part (b) of the theorem we follow exactly the same technique with $j_1 = j_2 = 1, m1 = n$ and $m_2 = m$. The bounds of the simple integrals are taken as in proof of Theorem 4.1 of the paper of Böttcher and Grudsky [1].

c) The proof of the part (c) depends on the proof of part (a) as has been described in [1]. $\bullet$

We can combine the above two Theorems to obtain results for the condition number. First we suppose that $n \sim m \sim \nu$, since that is the only case with practical importance. If the hypotheses of Theorem 2.1 (where the roots would be roots of the whole function f and not only of u) are holding and, for every root of f, the hypotheses of part (a) of Theorem 2.2 are also holding, then we can obtain results of the order of the condition number. From (a) of Theorem 2.2 we get that

$$\kappa(T_{nm}(f)) \geq C_1 \left(\frac{n^{\alpha_1} m^{\alpha_2}}{n^{\alpha_1} + m^{\alpha_2}} \right) \sim \nu^{\alpha},$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$, which corresponds to the maximum value of α's over all the roots. On the other hand Theorem 2.1 gives us that

$$\|T_{nm}^{-1}(f)\| < 12(\|v\|_\infty + 1)\omega(c(n+m)) = C_2\omega(\nu) = \mathcal{O}(\nu^{\alpha}).$$

So,

$$\kappa(T_{nm}(f)) \sim \nu^{\alpha}.$$

Analogous results can be obtained in the cases of logarithmic or exponential orders of roots or in the cases of mixed ones.

2.4 Some specialized results for the Hermitian case

In this subsection we discuss in more detail the Hermitian case (weakly sectoriality of the symbol f with null imaginary part). We present two results, a negative one and a positive one.

In the negative one it is shown that rotations and dilations of the domain lead to a substantial change in the condition number of finite sections of Toeplitz matrices so that the conditioning of $T_{nm}(f(x-y, x+y))$ cannot be reduced to the one of $T_{nm}(f(x,y))$: more precisely, we furnish an example

Table 1: The minimum eigenvalue of $T_{nm}(f)$ with $f(x,y) = (2-2\cos(x)) + (2-2\cos(y))^2$

n	m	N	$\lambda_{m^3m} \equiv \lambda_{\min}(T_{m^3m}(f))$	$\log(\lambda_{m^3m}/\lambda_{8m^32m})$
4^3	4	256	0.39679	-
8^3	8	4096	0.05057	2.97
16^3	16	65536	0.0047839	3.4
32^3	32	1048576	0.00028	3.83

Table 2: The minimum eigenvalue of $T_{nm}(f \circ U)$ with $(f \circ U)(x,y) = (2 - 2\cos(x-y)) + (2-2\cos(x+y))^2$

n	m	N	$\lambda_{m^3m} \equiv \lambda_{\min}(T_{m^3m}(f))$	$\log(\lambda_{m^3m}/\lambda_{8m^32m})$
4^3	4	256	0.78799	-
8^3	8	4096	0.17679	2.15
16^3	16	65536	0.04082	2.11
32^3	32	1048576	0.01121	1.87

where the condition numbers of $T_{nm}(f)$ and $T_{nm}(f \circ U)$ have asymptotically different growth rates with the polynomial $f(x,y) = (2-\cos(x)) + (2-\cos(y))^2$ and with $U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ being $\sqrt{2}$ times a 2-by-2 rotation matrix.

As displayed in Table 2.4, it is evident that the minimal eigenvalue of $T_{m^3m}(f)$ behaves as m^{-4} and this agrees with the general estimate $n^{-2} + m^{-4}$ when $n = m^3$ (combine Theorems 2.1 and 2.2 or refer directly to Theorem 2.3). In the second case, under the same assumptions on the partial dimensions ($n = m^3$), we observe that the minimal eigenvalue of $T_{m^3m}(f \circ U)$ behaves as m^{-2} (all the computations were carried out in Matlab with stopping criterion 10^{-5}). Therefore, since in both the examples, the maximal eigenvalues converge to $20 = \|f\|_\infty = \|f \circ U\|_\infty$ as $n, m \to \infty$, it follows that $\kappa(T_{m^3m}(f \circ U))$ and $\kappa(T_{m^3m}(f))$ have different order of magnitude as m tends to infinity.

Concerning the positive result, under very mild assumptions on the zeros of $f = \mathrm{Re} f$ and with the help of the notion of linear positive operator, it is proved that a lower bound for the condition number can be easily obtained by tensor arguments and the one level results in [1]. More in details the following results hold.

Theorem 2.3 Let $f \in L^\infty$ be a 2-variate, 2π-periodic, weakly sectorial function of additively separable type i.e. $f(x,y) = g(x) + h(y)$ for g and h being L^∞ and 2π-periodic. If $g, h \geq 0$ then

$$\max\{\lambda_{\min}(T_n(g)), \lambda_{\min}(T_m(h))\} \leq \lambda_{\min}(T_{nm}(f)) \leq \lambda_{\min}(T_n(g)) + \lambda_{\min}(T_m(h)) \tag{20}$$

and therefore there exist positive constants C_1, C_2, c_1, and c_2 such that

$$C_1 \frac{\omega_g(c_1 n)\omega_h(c_1 m)}{\omega_g(c_1 n) + \omega_h(c_1 m)} \leq \kappa(T_{nm}(f)) \leq C_2 \min\{\omega_g(c_2 n), \omega_h(c_2 m)\} \tag{21}$$

with

$$\frac{1}{2}\min\{\omega_g(cn), \omega_h(cm)\} \leq \frac{\omega_g(cn)\omega_h(cm)}{\omega_g(cn) + \omega_h(cm)} \leq \min\{\omega_g(cn), \omega_h(cm)\}$$

with $\omega_g(\cdot)$ being as in Theorem 3.4 and Eq. (6) of [1] (the one level version of our two level objects in Theorem 2.1 and Eqs. (1)–(2)).

Proof: It is a simple manipulation of the one level results and of the tensor structure of $T_{nm}(f) = T_n(g) \otimes I_m + I_n \otimes T_m(h)$ with I_k denoting the identity of size k. In fact, by the nonnegativity of g and h we deduce that

$$T_{nm}(f) \geq T_n(g) \otimes I_m, \quad T_{nm}(f) \geq I_n \otimes T_m(h)$$

and therefore

$$\lambda_{\min}(T_{nm}(f)) \geq \lambda_{\min}(T_n(g) \otimes I_m) = \lambda_{\min}(T_n(g)),$$
$$\lambda_{\min}(T_{nm}(f)) \geq \lambda_{\min}(I_n \otimes T_m(h)) = \lambda_{\min}(T_m(h)).$$

The latter joint with the one level results in [1] implies the left inequalities in (20) and (21). For the right inequalities it is enough to recall that

$$\lambda_{\min}(T_{nm}(f)) \leq \mathbf{v}^H T_{nm}(f)\mathbf{v}, \quad \forall \mathbf{v} : \ \|\mathbf{v}\|_2 = 1$$

and to consider the special vector $\mathbf{v} = \mathbf{x} \otimes \mathbf{y}$ where $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$ and where

$$\lambda_{\min}(T_n(g)) = \mathbf{x}^H T_n(g)\mathbf{x}, \quad \lambda_{\min}(T_n(h)) = \mathbf{y}^H T_n(h)\mathbf{y}.$$

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3 Conclusions

We have shown that the upper bound of the condition number depends on the maximal order of the zeros of Ref: the presence of a large Im f influences the extremal behavior by decreasing the condition number. Tight lower estimates have been found when f is real valued. Our analysis includes the case of symbols with a finite number of curves of zeros as well. We stress that this case is not trivial since it cannot be reduced by tensor arguments to the one level case as it easily happens when the symbol has only isolated zeros.

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